Oscillations in Systems with Non-Linear Reactance

By R. V. L. HARTLEY

A theoretical study is presented of the properties of a condenser, one plate of which is free to vibrate, when it is included in a circuit containing a generator, the frequency of which is higher than the resonant frequency of the plate and unrelated thereto. It is shown that the plate may be maintained in oscillation at a frequency at or near its mechanical resonance, at the expense of the energy supplied by the generator, provided certain conditions are satisfied. The most favorable condition is one in which the plate is resonant at the frequency of its vibration and the electric circuit is resonant at that of the generator, and at the difference between the generator and plate frequencies, and is anti-resonant at their sum. Under these conditions the generator voltage must exceed a threshold value determined by the impedances and frequencies. This threshold voltage increases as the conditions become less favorable. Expressions are given for the values of the oscillations as functions of the voltage when the threshold is exceeded. When the sum frequency is absent, the energies dissipated at the plate and difference frequencies are in the ratio of the two frequencies.

The oscillations described represent a special case of a class of similar oscillations, all of which depend on the presence of a non-linear reactance. Another special case is a molecular model capable of reproducing the main features of the Raman effect.

Introduction

A TYPE of free oscillation has been found to occur in non-linear coupled systems, which differs from the ordinary type in that the supporting energy is drawn from an alternating sustained source, rather than from a constant source, as in the ordinary vacuum tube oscillator. The particular example of such oscillations to be described here occurs in an electric circuit containing a condenser, one plate of which is elastically supported so as to constitute a mechanically resonant system.

The possibility of such oscillations in a circuit of this kind was discovered [1] in the course of a theoretical study of the possible use of a moving plate condenser as a modulator in a carrier system. Such use was suggested by the fact that, in a condenser, the mechanical force on the plate is proportional to the square of the charge. In this study, it was assumed that a generator of alternating electromotive force of a relatively high carrier frequency was connected with the condenser terminals, and an alternating mechanical force of a relatively low frequency (corresponding to a Fourier component of a speech wave) was applied to the movable plate of the condenser. The plate was not assumed to be resonant. The non-linear relations between

[1] Hartley; *Phys. Rev.*, Vol. 33, p. 289, February, 1929.

charge and mechanical displacement then give rise to currents of the combination, or sideband frequencies. Among the properties of the system which were studied was the reaction of the plate on the mechanical "generator." This was expressed as a mechanical impedance, i.e., the complex ratio of the alternating force to the alternating velocity.

The expression for this mechanical impedance was found to include a negative resistance, which under certain conditions became equal to the positive resistance representing the remainder of the system. It was evident, therefore, that, under these conditions, oscillations of the frequencies involved could persist in the absence of any external driving force on the plate. The existence of such oscillations was first verified experimentally by Mr. E. Peterson. This and a quantitative experimental study of the phenomenon are described in an accompanying paper.[2] Oscillations of the same general type, associated with iron core coils, had been predicted much earlier by the writer and discovered independently by Mr. E. T. Burton.[3]

However, what happened once the threshold condition was passed, was not apparent from this analysis. The answer to this question was found by assuming the existence of the oscillations, computing their values, and determining under what conditions the values are real. Both methods will be employed in what follows.

Representation of the System

In the analysis it will be assumed that, except for the non-linearity associated with the electromechanical coupling, the law of superposition holds throughout. This means that all parts of the system other than the coupling may be represented by linear impedances, of the form

$$Z = R + iX = Ze^{i\varphi}. \tag{1}$$

"Linear," as here used, means that the impedance is independent of the magnitudes of the oscillations.

If then the plate has an alternating velocity of magnitude V_m and phase θ_m, we represent it by $V_m e^{i\theta_m}$. The resultant of all the linear restoring forces may be represented by a force $Z_m V_m e^{i(\varphi_m+\theta_m)}$. All of the quantities involved will, in general, be functions of the frequency. Similarly a current $I_e e^{i\theta_e}$ will be accompanied by a counter electromotive force $Z_e I_e e^{i(\varphi_e+\theta_e)}$, where Z_e is the impedance of the connected electric circuit in series with that of the condenser with its movable plate at rest in the position of zero displacement.

[2] Hussey, L. W. and Wrathall, L. R.; "Oscillations in an Electromechanical System" in this issue of the *Bell Sys. Tech. Jour.*

[3] Peterson, E.; *Bell Laboratories Record*, Feb., 1929, p. 231.

To evaluate the non-linear forces, consider a parallel plate, air condenser of area, A, and normal separation, x_0, one plate of which is fixed and the other of which is free to move under the action of linear restoring forces. Let the movable plate be displaced a distance, x, in a direction to increase the separation, and let a charge, q, be put on the plates. Then it can easily be shown that the static forces tending to oppose the displacements are

$$= Sx + Kq^2, \tag{2}$$

$$e = \frac{q}{C} + 2Kxq, \tag{3}$$

where S is the stiffness of the constraints on the plate; C is the capacitance, in electrostatic units, when x is zero; and

$$K = \frac{2\pi}{A} \tag{4}$$

is a quantity which will be referred to as the constant of non-linearity.

The first terms of (2) and (3) represent components of the forces which were represented above by the mechanical and electrical impedances respectively. Hence only the last terms need be used in expressing the electromechanical coupling.

We shall assume that there is connected in series with the condenser and its associated electric impedance, a generator of negligible internal impedance, which provides an alternating electromotive force, e_g, of amplitude, E_g, and frequency, ω_g, in radians per second. The phase of this generator will arbitrarily be taken as zero.

For the first part of the analysis, we shall assume that there is an alternating force, f_m, exerted on the plate by a "mechanical generator," which has an amplitude, F_m, frequency, ω_m, and phase, ψ_m. We shall investigate the impedance offered to this force in the resulting condition of forced oscillation. In the second part, the mechanical generator will be omitted, and the free oscillations investigated. It is first necessary, however, to determine what frequencies need be considered.

Possible Frequencies

With the system just described there will be developed oscillations, the frequencies of which constitute an infinite series. It will therefore be necessary to introduce limiting assumptions. First let us consider what frequencies may be present in the system. In doing this it must be recognized that the conventional use of complex quantities is not justified when the system is non-linear. This difficulty is avoided and the advantages of the complex exponential notation are retained

if we use the complete exponential expressions for the trigometric functions, since these are real.

Accordingly we shall call the electromotive force of the generator

$$e_g = \frac{E_g}{2}[e^{i\omega_g t} + e^{-i\omega_g t}]. \tag{5}$$

We shall assume that this is accompanied by an alternating current,

$$i_g = \frac{I_g}{2}[e^{i(\omega_g t+\theta_g)} + e^{-i(\omega_g t+\theta_g)}]. \tag{6}$$

We shall call the force exerted by the mechanical generator

$$f_m = \frac{F_m}{2}[e^{i(\omega_m t+\psi_m)} + e^{-i(\omega_m t+\psi_m)}], \tag{7}$$

and the accompanying alternating velocity

$$v_m = \frac{V_m}{2}[e^{i(\omega_m t+\theta_m)} + e^{-i(\omega_m t+\theta_m)}]. \tag{8}$$

When the corresponding displacements, obtained by integration of (6) and (8), are substituted in the last term of (3), the resulting electromotive force is found to consist of components of frequencies,

$$\omega_s = \omega_g + \omega_m, \tag{9}$$

$$\omega_d = \omega_g - \omega_m, \tag{10}$$

which tend to set up currents at the frequencies of the sidebands.

If such currents flow and we substitute the charges associated with them, together with that from (6), in the last term of (2), we find, in the force on the plate, components of frequency ω_m, and a variety of other frequencies including zero, i.e., a steady force. If these produce displacements which are again substituted in (2), and the process is continued, we arrive finally at the entire series of frequencies given by $m\omega_g \pm n\omega_m$, where m and n are integers.

We shall now introduce the limiting assumption that the plate is resonant at or near ω_m, and not at any other frequency. The impedance at that frequency will then be small and the response to the driving force at that frequency relatively large. At the frequencies of all the other components of the force the mechanical impedance will be relatively very high; and we will not be making a violent assumption if we say that it is so high that the velocities of response at all the other frequencies are negligible. [There may be some response to the steady force, consisting of a slight change in the position of equilibrium about which the vibrations occur. This can be taken care of by saying that the coefficients in (2) and (3), while constant for any

particular condition of sustained oscillation, vary slightly with the magnitude of the oscillations.] With this assumption the frequencies of the components of the electromotive force reduce to $\omega_g \pm n\omega_m$.

If the electric circuit is resonant at any of these frequencies, we may as above neglect the currents at other frequencies. In the absence of any resonance, if the constant of non-linearity, K, is sufficiently small, the amplitudes will decrease so rapidly with increasing n that we may neglect all for which n is greater than unity. In the interests of simplicity we shall make all of these assumptions. We have then in addition to i_g and V_m the currents,

$$i_s = \frac{I_s}{2}\left[e^{i(\omega_s t+\theta_s)} + e^{-i(\omega_s t+\theta_s)}\right], \tag{11}$$

$$i_d = \frac{I_d}{2}\left[e^{i(\omega_d t+\theta_d)} + e^{-i(\omega_d t+\theta_d)}\right]. \tag{12}$$

Forced Oscillations; Impedance Solution

We wish to set up the equations of motion in terms of the applied forces, the velocities and currents at the various frequencies, and the properties of the system, as expressed in terms of its linear impedances and the constant of non-linearity, K. For each frequency, we equate whatever applied force there may be to the sum of the restoring forces due to the system. These consist of a component given by the product of the velocity or current by the impedance for the particular frequency, and other terms due to the combination of pairs of the other frequencies. To find these latter components, we integrate (6), (8), (11) and (12) with respect to time, substitute the resulting displacements in the last terms of (2) and (3), and select the components of the four significant frequencies, for insertion in their appropriate equations. Once these components are obtained, we may, since the remainder of the system is linear, safely revert to the conventional use of exponentials, so that the factor $e^{i\omega t}$ may be divided out for each equation. The final result is

$$Z_g I_g e^{i(\theta_g+\varphi_g)} + \frac{KV_m I_s}{\omega_m\omega_s} e^{i(\theta_s-\theta_m)} - \frac{KV_m I_d}{\omega_m\omega_d} e^{i(\theta_m+\theta_d)} = E_g, \tag{13}$$

$$Z_m V_m e^{i(\theta_m+\varphi_m)} + \frac{KI_g I_s}{\omega_g\omega_s} e^{i(\theta_s-\theta_g)} + \frac{KI_g I_d}{\omega_g\omega_d} e^{i(\theta_g-\theta_d)} = F_m e^{i\psi_m}, \tag{14}$$

$$Z_s I_s e^{i(\theta_s+\varphi_s)} - \frac{KI_g V_m}{\omega_g\omega_m} e^{i(\theta_g+\theta_m)} = 0, \tag{15}$$

$$Z_d I_d e^{i(\theta_d+\varphi_d)} + \frac{KI_g V_m}{\omega_g\omega_m} e^{i(\theta_g-\theta_m)} = 0. \tag{16}$$

These equations are of the second degree and so are not so simple of solution as are the linear equations of circuit theory which they formally resemble. We note, however, that if, in the last three, we assume I_g and θ_g to be constant, they become linear. We may therefore solve them as linear equations, with this assumption, provided we bear in mind that the resulting impedances will not be linear unless the oscillations are so small that the second and third terms of (13) can be neglected compared with the first.

Let us make this assumption and explore the properties of the resulting linear system represented by (14), (15) and (16). If we calculate $V_m e^{i\theta_m}$ and take the ratio $F_m e^{i\psi_m}/V_m e^{i\theta_m}$, this will be the analog of the impedance of an analogous electric circuit as measured in the mesh corresponding to vibration of the plate at frequency ω_m. This ratio, which we shall call Z_m', may be thought of as the mechanical impedance of the plate when the circuit is activated by the electrical generator. Following circuit theory, as applied to vacuum tubes, let us call Z_m' the active impedance of the plate, and Z_m the passive impedance. The value of the active impedance, when expressed in terms of resistances and reactances, is found to be

$$Z_m' = (R_m + iX_m) + \frac{K^2 I_g^2}{\omega_g^2 \omega_m \omega_s} \cdot \frac{R_s - iX_s}{Z_s^2} + \frac{K^2 I_g^2}{\omega_g^2 \omega_m \omega_s} \cdot \frac{-R_d - iX_d}{Z_d^2}. \quad (17)$$

We see that the active impedance differs from the passive impedance by two terms, each of which represents the effect of the impedance of the electric circuit at one of the side frequencies. The second term of (17), which depends on the impedance at the sum frequency, is identical in form with the impedance added to an electric circuit,[4] at a frequency, ω, by a transformer of mutual inductance, M, provided that

$$M^2\omega^2 = \frac{K^2 I_g^2}{\omega_g^2 \omega_m \omega_s}; \quad (18)$$

the impedance of the secondary circuit is equal to Z_s; and the reactances of the primary and secondary windings are included in X_m and X_s, respectively. The third term which depends on the impedance of the electric circuit at the difference frequency, is similar except that the effective resistance is negative.

It is this negative resistance which makes possible the type of free oscillations here described. To interpret it, let us start with the small

[4] Bush, V.; "Operational Circuit Analysis," John Wiley & Sons, Inc., 1929, p. 50, Eq. (66).

applied force, f_m, acting on the plate, with the voltage of the electric generator zero. The velocity of the plate vibration is then determined by its passive impedance alone. Let us assume for the time being that the impedance, Z_s, of the electric circuit at the sum frequency is infinite, so that its effect on the active impedance of the plate disappears. Let us make φ_d equal to φ_m, and gradually increase the generator voltage. As I_g^2 increases, the negative impedance increases, the total impedance decreases and the velocity, V_m, increases. This condition is analogous with the behavior of the input impedance of a regeneratively connected amplifier when the plate current is progressively increased from zero. At a threshold value of I_g, the net impedance becomes zero and the velocity infinite. This means that a finite velocity can exist for an infinitesimal driving force, that is, the oscillations, once started, are self-sustaining, even in the absence of any sustained driving force, f_m, at the mechanical frequency.

If we make the electric impedance, Z_d, at the difference frequency infinite, all the resistances are positive; so sustained oscillations cannot occur, in a dissipative system, in the absence of current at the difference frequency. If both side frequencies are present, so that Z_s and Z_d are both finite, sustained oscillations are still possible provided the impedance at the sum frequency is not too small compared with that at the difference frequency. The presence of current at the sum frequency always increases the critical value of the current at the generator frequency.

We may also compute the active impedance of the electric circuit at the side frequencies, on the same assumption as to the constancy of the current of generator frequency as was made in deriving (17). To do this, we remove the mechanical generator, making the right member of (14) zero, and insert low measuring voltages of frequencies ω_s and ω_d in the right members of (15) and (16), in turn. In each case we compute the ratio of this voltage to the accompanying current. If we think of each frequency as being the analog of a mesh in an electric circuit, we note that the mesh corresponding to the mechanical frequency is coupled to both of the side frequencies; but the latter are not directly coupled to each other. If the mutual impedances, which depend on I_g, are small enough, we may, for a generator at the sum frequency, neglect the third term of (14), which represents the effect of the loosely coupled difference frequency mesh, compared with the first. The active impedance at the sum frequency then becomes

$$Z_s' = (R_s + iX_s) + \frac{K^2 I_g^2}{\omega_g^2 \omega_m \omega_s} \cdot \frac{R_m - iX_m}{Z_m^2}. \tag{19}$$

If the third term of (14) is not neglected we must replace Z_m in (19), by the first and third terms of (17), that is, by the impedance of the mechanical frequency mesh, as modified by its coupling with the difference frequency mesh.

Similarly, when the measuring generator is of the difference frequency, we get

$$Z_d' = (R_d + iX_d) + \frac{K^2 I_g^2}{\omega_g^2 \omega_m \omega_s} \cdot \frac{-R_m - iX_m}{Z_m^2}, \tag{20}$$

where Z_m is to be replaced by the first and second terms of (17), if the second term of (14) is not neglected.

The active impedance at the difference frequency (20) contains a negative resistance similar to that which appeared at the mechanical frequency (17). In fact, if the passive impedance, Z_s, at the sum frequency is infinite, the expressions for the two active impedances are symmetrical. The active impedance at the sum frequency contains only positive resistances, except in so far as the resistance of the mechanical mesh is made negative by its coupling with the difference mesh. This serves to emphasize the fact that the presence of current of the difference frequency is essential to the oscillations, while that of current of the sum frequency tends to make their production more difficult.

Free Oscillations

In the above considerations it was assumed that the amplitudes at all of the new frequencies were small compared with that at the generator frequency. While this assumption permits us to compute the threshold conditions for the starting of free oscillations, it is violated as soon as the oscillations become appreciable. In order to find out what happens once the threshold is passed it is necessary to solve the second degree equations (13) to (16) when F_m is made zero. The presentation of this solution will be simplified by considering first the case where the sum frequency is eliminated and then the effect of its presence on the simpler solution.

The elimination of the sum frequency is accomplished by making Z_s infinite and I_s zero. This makes the second terms of (13) and (14) zero, and makes (15) indeterminate. We are left then with (13), (14), as modified, and (16). The equations for the mechanical and difference frequencies are now symmetrical. In order to solve these equations we express the exponentials in terms of sines and cosines and equate the real and imaginary parts separately. In the equations derived from (14) and (16) we transpose the second term in each equation to the right member. For each pair we divide the equation containing sines by

that containing cosines and obtain a relation between the angles involved. We square each equation of a pair, and add them to obtain a relation between the magnitudes of velocities and currents, the impedances, and the frequencies. From these it follows that I_g is a constant. By means of these relations the equations derived from (13) may be reduced to a form where the only variables are E_g, V_m and θ_g, and the constant, I_g, appears only as a divisor of E_g. These equations are then squared and added to give an equation which determines V_m. The final solution takes the form

$$\varphi_m = \varphi_d, \tag{21}$$

$$I_g = \frac{\omega_g}{K}[Z_m\omega_m Z_d\omega_d]^{1/2}, \tag{22}$$

$$V_m = \frac{\omega_g}{K}\left[Z_d\omega_d Z_g\omega_g \left(-\cos(\varphi_m + \varphi_g) \pm \left\{ \left(\frac{E_g}{Z_g I_g}\right)^2 - \sin^2(\varphi_m + \varphi_d) \right\}^{1/2} \right) \right]^{1/2}, \tag{23}$$

$$I_d = \left[\frac{Z_m\omega_d}{Z_d\omega_m} \right]^{1/2} V_m, \tag{24}$$

$$\theta_g = \varphi_m + \alpha \pm \frac{\pi}{2}, \tag{25}$$

where

$$\cos\alpha = \frac{Z_g I_g}{E_g}\sin(\varphi_m + \varphi_g), \tag{26}$$

and the sign in (25) is so chosen that

$$-\frac{\pi}{2} < \theta_g < \frac{\pi}{2}, \tag{27}$$

and

$$\theta_m + \theta_d = \alpha + \pi \pm \frac{\pi}{2}, \tag{28}$$

where the same sign is to be taken for $\pi/2$ as in (25).

The nature of the variation represented by (23) is shown in Fig. 1, which is taken from the accompanying experimental paper.[2] Here the amplitude, V_m/ω_m, of the plate displacement is plotted against the generator voltage, E_g, for the case of exact resonance and for one involving a slight departure from resonance.

Let us interpret these results physically. The phase angles in (21) depend only on the physical constants of the system and the frequencies of the oscillations. This equation, therefore, determines at what frequencies oscillations may occur provided the other conditions are

satisfied. Thus the ratio of reactance to resistance must be the same for the plate at ω_m and for the electric circuit at ω_d. This condition is satisfied if each is resonant at its particular frequency, but resonance is not a necessary condition. All that is necessary is that there be a pair of frequencies, whose sum is equal to that of the electric generator, for which the impedances have the same phase angle. If there are an electric and a mechanical resonance such that the sum of the resonant frequencies is nearly equal to the generator frequency, and there is a marked difference in the sharpness of the two resonances, then the oscillations will fall closer to the sharper resonance. This is due to the fact that the phase angle of the impedance changes more rapidly with frequency in the neighborhood of a sharp resonance.

From (22) we see that the amplitude of the current at the generator frequency depends only on this frequency, the constants of the system, and the new frequencies. It is independent of the amplitude of the

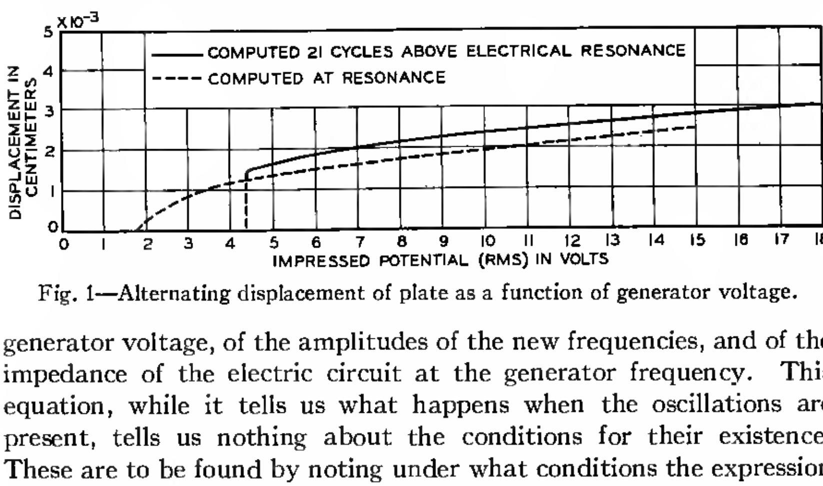

Fig. 1—Alternating displacement of plate as a function of generator voltage.

generator voltage, of the amplitudes of the new frequencies, and of the impedance of the electric circuit at the generator frequency. This equation, while it tells us what happens when the oscillations are present, tells us nothing about the conditions for their existence. These are to be found by noting under what conditions the expression (23) for the amplitude at the new frequency, ω_m, is real. We have two cases to consider which are determined by the sign of $\cos(\varphi_m + \varphi_g)$.

Assume first that this is positive, as would be the case if the plate is resonant at ω_m and there is any dissipation at ω_g, such as would be caused by resistance in the electric circuit. The first term in (23) is negative and V_m can be real only if the second term exceeds it in absolute value. This condition reduces to

$$E_g > Z_g I_g = \frac{Z_g \omega_g}{K} [Z_m \omega_m Z_d \omega_d]^{1/2}. \tag{29}$$

This shows that there is a threshold value of the generator voltage,

above which the new oscillations are possible. (It is found to agree with that obtained by the negative resistance method.) Moreover, this value is that which is just necessary to maintain an electric current, of the generator frequency, in the absence of the new frequencies, with an amplitude equal to the constant amplitude that exists in the presence of the new frequencies. For values of E_g large compared with the threshold value, the amplitudes of the new frequencies increase nearly as the square root of the amplitude of the generator voltage.

In the special case of resonance at both ω_m and ω_d, Z_m and Z_d tend to be small and so from (24) the threshold voltage is correspondingly small. This therefore is a particularly favorable condition for the production of the oscillations.

The case where $\cos (\varphi_m + \varphi_g)$ is negative occurs when all of the three impedances are predominantly reactive, the reactances being all of the same sign. The first term of (23) is then positive and V_m will be real if the second term is positive, as it will be if

$$E_g > Z_g I_g |\sin (\varphi_m + \varphi_g)|. \tag{30}$$

For this case, then, the threshold amplitude of the generator voltage may be much less than that required to maintain the current at the constant amplitude, I_g, in the absence of the new frequencies.

In the extreme case where there is no dissipation and the phase angles of the impedances are all $\pm \pi/2$, the threshold voltage reduces to zero and so sustained oscillations are possible in the absence of any generator. (23) and (24) then reduce to forms symmetrical with (22). This means that for such a system the frequencies would be determined by the constants of the system and the amount of energy present, since this would limit the possible amplitudes.

There is some question as to the sign to be given to the inner radical in (23). When $\cos (\varphi_m + \varphi_g)$ is positive the plus sign must be used. When it is negative the plus sign must be used if E_g is greater than $Z_g I_g$. If E_g is between this and the threshold given by (30), either sign gives a real value for the amplitude. When the sign is negative the amplitude decreases with increasing voltage, which appears to be an unstable condition.

Regarding the phases, the condition represented by (27) is imposed because the energy flow must be from the generator to the circuit. Only the sum of the phases of the new oscillations is determined. Their individual values depend on the starting conditions, just as does the phase of a pendulum clock.

One more result may be of interest. This is the relative rates at which energy is dissipated at the two new frequencies. If P_m and P_d

are the powers corresponding to the two frequencies we have

$$\frac{P_d}{P_m} = \frac{I_d^2 Z_d \cos \varphi_d}{V_m^2 Z_m \cos \varphi_m} = \frac{\omega_d}{\omega_m}. \tag{31}$$

Thus the rate of energy dissipation is in the ratio of the frequencies.

Effect of Sum Frequency

The more general case where the sum frequency is also present calls for the solution of (13) to (16) as they stand except for F_m being zero. This may be done by substituting the values of I_s and θ_s from (15), and those of I_d and θ_d, from (16), in (13) and (14), and proceeding in a manner similar to that used above. The results take the form

$$\varphi_m = \gamma, \tag{32}$$

where

$$\tan \gamma = \frac{Z_s\omega_s \sin \varphi_d + Z_d\omega_d \sin \varphi_s}{Z_s\omega_s \cos \varphi_d - Z_d\omega_d \cos \varphi_s}, \tag{33}$$

and the signs of $\sin \gamma$ and $\cos \gamma$ are determined by the numerator and denominator of (33) respectively;

$$I_g = \frac{\omega_g}{K}\left[\frac{Z_m\omega_m Z_d\omega_d}{a}\right]^{1/2}, \tag{34}$$

where

$$a = \left[1 + \left\{\frac{Z_d\omega_d}{Z_s\omega_s}\right\}^2 - 2\frac{Z_d\omega_d}{Z_s\omega_s}\cos(\varphi_d + \varphi_s)\right]; \tag{35}$$

$$V_m = \frac{\omega_m}{K}\left[\frac{Z_d\omega_d Z_g\omega_g}{b}\left(-\cos(\delta + \varphi_g) \pm \left[\left(\frac{E_g}{Z_g I_g}\right)^2 - \sin^2(\delta + \varphi_g)\right]^{1/2}\right)\right]^{1/2}, \tag{36}$$

where

$$b = \left[1 + \left(\frac{Z_d\omega_d}{Z_s\omega_s}\right)^2 + 2\left(\frac{Z_d\omega_d}{Z_s\omega_s}\right)\cos(\varphi_d - \varphi_s)\right]^{1/2}, \tag{37}$$

and

$$\tan \delta = \frac{Z_s\omega_s \sin \varphi_d + Z_d\omega_d \sin \varphi_s}{Z_s\omega_s \cos \varphi_d + Z_d\omega_d \cos \varphi_s}; \tag{38}$$

$$I_d = \left[\frac{Z_m\omega_d}{Z_d\omega_m a}\right]^{1/2} V_m; \tag{39}$$

$$I_s = \frac{Z_d}{Z_s} I_d; \tag{40}$$

$$\theta_g = \delta + \alpha' \pm \frac{\pi}{2}, \tag{41}$$

where

$$\cos \alpha' = \frac{Z_g I_g}{E_g} \sin (\delta + \varphi_g); \qquad (42)$$

$$\theta_m + \theta_d = \alpha' + \pi \pm \frac{\pi}{2} + (\delta - \varphi_d); \qquad (43)$$

$$\theta_s - \theta_m = \alpha' \pm \frac{\pi}{2} + (\delta - \varphi_s), \qquad (44)$$

where the sign of $\pi/2$ is again to be chosen so as to satisfy (27).

Corresponding to (21) we have (32). If the mechanical motion involves any dissipation, the mechanical resistance, $Z_m \cos \varphi_m$, must be positive, and since Z_m is positive by definition, $\cos \varphi_m$ must be positive. This means that (32) can be satisfied only if the denominator of (33) is positive. Hence oscillations can occur only if

$$\frac{Z_d \omega_d}{Z_s \omega_s} < \frac{\cos \varphi_d}{\cos \varphi_s}. \qquad (45)$$

This relation can hold when Z_d, the impedance at the difference frequency, is infinite, only if φ_s is $\pm\ \pi/2$, that is, if there is no dissipation at the sum frequency.

To investigate the relative rates of dissipation at the sum and difference frequencies, we find the ratio of the powers P_s and P_d, associated with them.

$$\frac{P_s}{P_d} = \frac{Z_d \cos \varphi_s}{Z_s \cos \varphi_d} < \frac{\omega_s}{\omega_d}. \qquad (46)$$

Thus the ratio is always less than the ratio of the frequencies and approaches it only as the limiting condition for oscillations is approached.

A discussion of all possible values of impedance and phase angle at the two side frequencies would be too involved to go into here. The special case of resonance at both frequencies is, however, of some interest since a given current is then accompanied by a maximum of dissipation. It also provides that ω_m coincides with the mechanical resonance, where Z_m is much smaller than for nearby frequencies. Since Z_m enters into the expression for the threshold force, this condition is particularly favorable for the occurrence of oscillations. When we make φ_d and φ_s zero we see from (45) that the impedances, now pure resistances, must be such that

$$\frac{Z_d \omega_d}{Z_s \omega_s} < 1. \qquad (47)$$

(35) now becomes

$$a = 1 - \frac{Z_d \omega_d}{Z_s \omega_s}. \tag{48}$$

From (34) it is evident that when Z_s is finite the constant current of the generator frequency, and so also the threshold voltage, are greater than when it is infinite. Thus the presence of current of the sum frequency makes the conditions for oscillation more exacting. As we approach the limiting impedance ratio, where the powers approach the ratio of their frequencies, the threshold voltage approaches infinity, and the probability of oscillations approaches zero.

The relative powers at the difference frequency and at the mechanical frequency are now given by

$$\frac{P_d}{P_m} = \frac{\omega_d}{\omega_m \left(1 - \dfrac{Z_d \omega_d}{Z_s \omega_s}\right)}. \tag{49}$$

The presence of a finite impedance at the sum frequency increases this ratio over that of the frequencies. For the limiting condition of oscillations it approaches infinity, the amplitude at the difference frequency then becoming infinite and that at the mechanical frequency remaining finite.

From these results it appears that proportionality between power and frequency is a limiting case which occurs only under the conditions which are most and least favorable for the existence of oscillations. We should, therefore, expect to find it only under the favorable conditions where the transformation of energy is from a higher to a pair of lower frequencies.

Effect of Other Frequencies

In the interests of simplicity the above treatment was limited to the case where all but four frequencies are suppressed by high impedances. Such a limitation is not, however, essential to the production of oscillations. In fact, as many as desired of the series $m\omega_g + n\omega_m$ may be produced by the proper choice of impedances and the use of high enough voltages, provided, of course, the apparatus can withstand the stresses involved. In general, the presence of certain frequencies will be favorable to oscillations and that of others unfavorable.

Summary

By way of summary, then, it is possible to maintain a movable condenser plate in sustained oscillation by applying to the condenser an

alternating electromotive force of an unrelated higher frequency, provided that the impedances of the system at these two frequencies and their various combinations satisfy certain relations, and the applied electromotive force exceeds a threshold value. When the oscillations are negligible at all frequencies except these two and their sum and difference, the most favorable condition, lowest threshold voltage, occurs when the plate vibrates at its resonant frequency, and the electric circuit is resonant at the applied frequency and at the difference frequency, and anti-resonant at the sum frequency. Once the oscillations start, the current of the applied frequency remains constant with increasing voltage. Under the most favorable conditions the rates of energy dissipation at the plate and difference frequencies are in the ratio of the frequencies.

Other Applications; Raman Effect

While in the case considered above the production of oscillations was associated with a particular type of non-linearity, the application of the principle is much more general. Here the non-linearity occurs in what might be called a mutual stiffness, serving to couple two degrees of freedom. It is not essential, however, that the non-linearity occur in a mutual impedance nor that the impedance be of the stiffness or negative reactance type. So long as the connected system is such as to provide the proper impedances, oscillations may occur in connection with any non-linear reactance.

A non-linear reactance, as here used, may be defined as any energy-storing element in which the coefficient of inertia is a function of the velocity, or that of stiffness is a function of the displacement, or any mechanical, electrical or electromechanical analog, of such an element. For a non-linear inertia, as in an iron core inductance coil, however, the power varies inversely as the frequency; instead of directly as for a non-linear stiffness.

A special case, in which one of the new frequencies is an exact submultiple of the driving frequency, has been studied by a number of workers from Rayleigh [5] down to Pedersen.[6]

Another special case may be of some interest to physicists because it provides a model of the Raman effect. The transition from the condenser to the molecular model will be made in two steps. For the first suppose that instead of making the resonant mechanical member one plate of a condenser, we attach the moving part to a point on its support by an elastic string under tension, the direction of the string

[5] Rayleigh; "Theory of Sound," *Sec. Ed.*, Vol. 1, p. 81.
[6] Pedersen; *Jr. Acous. Soc. Amer.*, Vol. VI, 4, p. 227, April, 1935.

being parallel to the direction of vibration. Suppose now that at some point along the string we apply an alternating mechanical force, acting normal to the string, through the medium of a mechanical structure, which as viewed from the string may be represented by a linear mechanical impedance. This structure prevents motion of its point of attachment to the string, in the direction of the string.

If now we analyze the forces and motions into their components in the direction, x, of the motion of the vibrating member and that, y, of the applied force, we find that, to a first approximation, the relations connecting them are identical with those used above for the condenser, provided we identify the force and velocity of the vibration in the x direction with those of the condenser plate and those of the point of attachment of the string, in the y direction, with the electromotive force and current in the electric circuit associated with the condenser. Such a structure can therefore produce oscillations of the sort described, provided the mechanical impedance of the driving structure has the proper values at the sum and difference frequencies.

Suppose now we have a molecule which we assume to be rigid with the exception of one atom, which is bound to it by a pair of electrons. Let the attached atom correspond to the plate, the relatively heavy molecule to the support and the electrons to the point of application of the driving force. Let the forces of electrostatic attraction between the electrons and the atom, and between the electrons and the center of the molecule, correspond to those due to the tense strings. Let the other static forces between the atom and the molecule correspond to the stiffness of the plate. For small displacements these forces may be assumed to vary linearly with distance, and so be capable of representation by constant coefficients of stiffness which correspond to the elasticities in the mechanical system. The applied external force is that exerted on the electrons by that component of the incident light which is normal to the line through the centers of the undisplaced particles. The mechanical impedance of the electrons for motion in the direction of the applied force corresponds to that of the structure through which the force is applied. This impedance includes the effects of any elastic constraints the rest of the molecule may exert on the electrons in this direction; of the electromagnetic mass of the electrons, which may be affected by the reactions of neighboring molecules; and of the dissipation of energy as radiation or by transfer to neighboring molecules.

Unlike other classical models of the Raman effect, this one provides for the persistence of the difference line, and the disappearance of the sum line, at low temperatures. It also provides that the intensity of

the lines should depend on the probability that the force exerted by the incident radiation on the electrons of a randomly chosen molecule exceed a threshold value which is determined by the condition of its neighbors. The apparent smallness of this probability would explain the observed weakness of the Raman lines.

It would seem that this threshold, and the probability of its being exceeded, might prove helpful in interpreting the energy threshold and transition probability which are used in wave mechanics.

Acknowledgment

In conclusion, I wish to express my thanks to those of my colleagues who by discussions and suggestions have contributed to the preparation of this paper, and in particular to Mr. L. A. MacColl for some valuable suggestions on the mathematical side.